Homotheties of a Class of Spherically Symmetric Space-Times Admitting G_3 as Maximal Isometry Group

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The homotheties of spherically symmetric spacetimes admitting G_4, G_6 and G_{10} as maximal isometry groups are already known, whereas for the space-times admitting G_3 as isometry groups, the solution in the form of differential constraints on metric coefficients requires further classification. For a class of spherically symmetric space-times admitting G_3 as maximal isometry groups without imposing any restriction on the stress-energy tensor, the metrics along with their corresponding homotheties are found. For the one case the metric is found along with its homothety vector that satisfies an additional constraint and is illustrated with the help of an example of a metric. For another case the metric and the corresponding homothety vector are found for a subclass of spherically symmetric space-times for which the differential constraint is reduced to separable form. Stress-energy tensor and related quantities of the metrics found are given in the relevant section.

I. INTRODUCTION

General Relativity (GR) [1] formulates a physical problem in terms of differential equations as a geometric requirement that a space-time may correspond to a Riemmannian manifold as the interaction of matter and gravitation. A relation between the geometry and the distribution of matter in the space-time is expressed by the following Einstein's field equations ($EFEs$) (1),

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} \,=\, \kappa T_{ab}. \tag{1}$$

GR is expressed in terms of Pseudo-Riemannian Geometry. The torsion free space (V_4) is represented by a Riemannian manifold M of four dimensions having signature (+, -, -, -) with metric tensor g_{ab} and symmetric stress energy tensor T_{ab} ($a, b = 0, 1, 2, 3$). The curvature of the space-time is represented by the Riemann tensor R_{abcd}, where $R_{ab} = R^{c}_{acb}$ is the contraction of Riemann curvature tensor, $R = R^{a}_{a}$, the Ricci scalar, $\kappa = 8\pi G/c^4$ and Λ, the cosmological constant. The value of Λ is observed to be negligible and usually taken to be zero. The Λ term is only of significance at cosmological scale. In case of non-vanishing Λ, the Λ-term is usually treated as part of the stress-energy tensor T_{ab}.

EFEs (1) break down into highly non-linear, second order coupled partial differential equations and are difficult to handel in general unless certain symmetries are assumed by the space-times. Exact solutions [2] of EFEs (1) may be found by requiring certain symmetry property of a space-time and they have played a significant role in the discussion of physical problems e.g. the Kerr and Schwarzschild solutions for the final collapsed state of massive bodies. The exact solutions mostly arose from highly idealized physical problems requiring high symmetry as has been compiled by Kramer et al [2], for example the well known spherically symmetric solutions of Schwarzschild, Reissner and Nordstrom, Tolman and Friedmann. The known exact solutions may be classified into (at least) four classes [2], namely the algebraic classification of conformal curvature, physical characterization of the energy momentum tensor, existence and structure of preferred vector fields and group of motions. The groups of symmetries is used to construct more general cosmologies. One of these symmetries called homotheties of a space-time are more restrictive than the isometries of the space-time. They are useful to find the solutions of $EFEs$, their properties and they can model the universe to find new facts related to cosmology and singularities [3]. Classical Hydrodynamics also has benefitted from the similarity solutions assuming the models for physical systems having no intrinsic scale of length, mass or time [3]. Cahill and Taub [4] analysed the homotheties of the spherically symmetric distribution of self-gravitating perfect fluid satisfying the homothety eqs. (4). Taub [5] studied the homotheties of plane symmetric space-times underlining the physical significance of homotheties in GR. Godfrey [6] constructed all homothetic Wyel space-times. Collinson and French [7], Katzin, Lavine and Davis [8] and Collinson [9] studied more general geometric symmetries. Farid, Qadir and Ziad [10] classified static plane symmetric spacetimes according to their Ricci collineations (RCs) and their

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relation with isometries of the space-times. Sharif and Sehar [11, 12] studied kinematic self-similar solutions of plane and cylindrically symmetric space-times for the perfect fluid and dust. Saifullah and Yazdan [13] studied conformal motions in the context of plane symmetric static space-times.

As mentioned above, one of such restrictions could be to allow a space-time to admit certain symmetry properties. These symmetry properties lead a space-time to obey a certain Lie group or an isometry group. The isometry group G_m of (M, g) is the Lie group of smooth maps of M into itself, leaving g invariant. The subscript m is equal to the number of generators or isometries of the group. It is the Lie algebra of continuously differentiable transformations $K = K^a\,(\partial/\partial x^a\,)$, where $K^a = K^a(x^b)$ are the components of the vector field K, known as a Killing vector (KV) field. A Killing vector field K is a field along which the Lie derivative of the metric tensor g is zero. i.e

$$\underset{K}{\mathscr{L}} g_{ab} = 0, \tag{2}$$

where $\mathscr{L}$ denotes the Lie derivative. Besides isometries, there are symmetries called the self-similar solutions of space-times which are more restrictive. These symmetry properties require a space-time to admit a Lie group, for example, the conformal motions, homothetic motions, Ricci collineations, curvature collineations, affine collineations etc. A homothety vector field $H = H^a(\partial/\partial x^a)$ is a field along which the Lie derivative of a metric tensor of a space-time remains invariant up to a scale, given by

$$\underset{H}{\mathscr{L}} g_{ab} = 2\phi_0 g_{ab}, \tag{3}$$

where ϕ_0 is a scalar parameter, called the homothetic constant and H the homothetic vector field. Above eq. (3) may be rewritten in the component form given below,

$$H^c\nabla_c g_{ab} + g_{ac}\nabla_b H^c + g_{bc}\nabla_a H^c = 2\phi_0 g_{ab}. \tag{4}$$

The corresponding homothety group is denoted by H_r, the subscript r is the number of generators of the group. For $\phi_0 = 0$, the homotheties become motions but the converse may not be true.

It is well known that for a Riemannian space V_n, the maximal group of motions is of the order less than or equal to $n\,(n + 1)\,/2$. Fubini [14] has proved that a Riemannian manifold V_n cannot admit a maximal group of the order $n\,(n+1)\,/2-1$. Yegorov [15] proved a result for Lorentzian manifolds, according to which the maximum group of mobility cannot be of the order$n\,(n+1)\,/2-2$. It is well known [3] that for a Riemannian manifold with metric g_{ab} and admitting G_m as the maximal group of isometries, H_r could be at the most of the order $r = m+1$. Thus for a V_n, H_r could be at the most of the order $r = n(n+1)/2+1$. The results of Fubini and Yegorov show that a spacetime V_n cannot admit a homothety group H_r with $r = n(n+1)/2-i$, where $i = 0, 1$. A detailed discussion of general relationship between isometries and homothetic motions can be seen in the work [16, 17]. It is known that for a V_n, there could be at the most $[n\,(n+1)\,/2] +\ 1$ homothetic motions. For the spherically symmetric space-times

$$ds^2 = e^{\nu(t,r)}dt^2 - e^{\lambda(t,r)}dr^2 - e^{x(t,r)}d\Omega^2, \tag{5}$$

$d\Omega^2 = d\vartheta^2 + \sin^2\vartheta\, d\phi^2$, it is known [18] that the spherically symmetric space-times cannot admit a G_5 as the maximal group of motions. Qadir and Ziad [19] proved that the spherically symmetric space-times allow isometry group G_m of dimension $m = 3, 4, 6, 7, 10$. Therefore these space times could admit homothety groups H_r of dimension $r = 4, 5, 7, 8, 11$. To find which space-time admits a non trivial homothety, the authors [20] solved the homothety equations for the spherically symmetric space-times for all the possible cases within the aforementioned class of space-times. It came out that $r \neq 8$. Thus, for spherically symmetric space-times the possible maximal homothety groups H_r could be of the order $r = 4, 5, 7, 11$. The solution of the homothetic eqs. (4), as discussed in ref. [20] results in the possible metrics along with the homothety groups H_5, H_7 and H_{11}. However, for the space-times admitting G_3 as the maximal isometry group the solution of the homothety eqs. (4) is provided in the form of derivatives of unknown metric coefficients, which then requires a further classification for the case of homothety group H_4.

In order to find the space-times along with their respective homothety groups, one needs to solve the eqs. (4) for the spherically symmetric space-times (5); for details and background we refer the reader to [20]. For $x\,(t, r) = \mu\,(t, r) + 2\ln r$, the solution of the homothety eqs. (4) for the spherically symmetric space-times (5) (dot and dash below denote derivatives $w.r.t.$ t and r, respectively) is given by:

$$H = H^0\frac{\partial}{\partial t} + H^1\frac{\partial}{\partial r} + H^2\frac{\partial}{\partial\vartheta} + H^3\frac{\partial}{\partial\varphi}, \tag{6}$$

where

$$
\begin{aligned}
H^0 &= -r^2 e^{\mu-\nu}\left(\sin\vartheta\left(\dot{g}_1\sin\varphi-\dot{g}_2\cos\varphi\right)+\dot{g}_3\cos\varphi\right)+g_4, && (7)\\
H^1 &= r^2 e^{\mu-\lambda}\left(\sin\vartheta\left({g'}_1\sin\varphi-{g'}_2\cos\varphi\right)+{g'}_3\cos\vartheta\right)+g_5, && (8)\\
H^2 &= -\cos\vartheta\left(g_1\sin\varphi-g_2\cos\varphi\right)+g_3\sin\vartheta+\left(c_1\sin\varphi-c_2\cos\varphi\right), && (9)\\
H^3 &= -\cos ec\vartheta\left(g_1\cos\varphi+g_2\sin\varphi\right)+\cot\vartheta\left(c_1\cos\varphi+c_2\sin\varphi\right)+c_3, && (10)
\end{aligned}
$$

where c_j $(j = 1, 2, 3)$ correspond to the generators of $SO(3)$ and g_k for $k = 1, 2, 3, 4, 5$ are functions of t and r, and they satisfy the following constraints:

$$
\begin{aligned}
-\dot{x}e^{x-\nu}\,\dot{g}_j+x'e^{x-\lambda}\,{g'}_j+\;2g_j &= 0, && (11)\\
2\ddot{g}_j+(2\dot{x}-\dot{\nu})\dot{g}_j-\nu'e^{\nu-\lambda}+2{g'}_j &= 0, && (12)\\
2\dot{g}'_j+(x'-\nu')\dot{g}_j+(\dot{x}-\dot{\lambda}){g'}_j &= 0, && (13)\\
2{g''}_j+(2x'-\lambda'){g'}_j-e^{\lambda-\nu}\dot{g}_j &= 0, && (14)\\
\dot{x}g_4+x'g_5 &= 2\varphi_0, && (15)\\
2\dot{g}_4+\dot{\nu}g_4+\nu'g_5 &= 2\varphi_0, && (16)\\
e^{\nu}{g'}_4-e^{\lambda}\dot{g}_5 &= 0, && (17)\\
2{g'}_5+\lambda'g_5+\dot{\lambda}g_4 &= 2\varphi_0. && (18)
\end{aligned}
$$

A complete solution of above eqs. (11)-(18) provides all possible metrics with their homotheties admitting homothety groups H_5, H_7, H_{11} except for homothety group H_4 for which the space-times should admit additional differential constraints. In [20], it is found that for $r = 5$ there are three spacetimes (2.1)-(2.3) admitting $G_4 \equiv SO(3)\otimes R$ where R is time-like, space-like and null respectively, includes all static spherically symmetric space-times and the Bertotti-Robinson metrics; for $r = 7$ there are four spacetimes (RobertsonWalker (RW) spacetimes (3.1) with (3.5) and a RW-like space-time (3.3) with (3.9)) which admit $G_6 \equiv SO(4), SO(3)\otimes R^3, SO(1,3)$ as the maximal isometry groups; for $r = 11$, the only spacetime is Minkowski space-time, for which the maximal group of isometries is $SO(1,3)\otimes R^4$; for $r = 4$, H_4 as the maximal group of homotheties the space-time admitting $G_3 \equiv SO(3)$ as the maximal isometry group satisfy additional differential constraints (4.3)-(4.7), which require further classification according to different types of stress-energy tensor as done by, e.g., Cahill and Taub [4].

In this paper, we find homotheties of a class of the spherically symmetric space-times (5) admitting $G_3 \equiv SO(3)$ as the maximal isometry group for $x(t,r) = 2\ln r$, imposing no restriction on the stress-energy tensor. We accomplish this task in the section II. This gives rise to the two cases that either $\dot{\lambda} = 0$ or $\dot{\lambda} \neq 0$. In the former case, the metric found is given by eq. (33) along with its homothety vector eq. (34). In particular for $\nu(t,r) = \frac{t}{r}$, $h(t) = t$, the corresponding metric and the homothety vector are given by the eqs. (35) and (36) respectively. In the latter case, the metric and the homothety vector are given by the eqs. (51) and (52), for a subclass of spherically symmetric space-times (5) for $\nu = \nu(r)$ for which one of the constraint equations is reduced to separable form. Furthermore, we have included Ricci tensor components R_{ab}, Ricci scalar R and the stress energy tensor T_{ab} of the space-times (35) and (51) in the relevant section. The results and remarks are presented in the final section III.

II. SPHERICALLY SYMMETRIC SPACE-TIMES ADMITTING H_4 AS THE HOMOTHETY GROUP

For the space-times eq. (5) to have $SO(3)$ as the maximal isometry group, one must have $g_j(t,r) = 0$, for which eqs. (11)-(14) are satisfied, where j = 1, 2, 3. However g_4, and g_5 satisfying the eqs. (15) - (18) should include only one arbitrary constant, the ϕ_0 corresponding to the scale parameter of the homothety. Let us suppose that,

$$
\begin{aligned}
g_4 &= \phi_0\, h(t,r), && (19)\\
g_5 &= \phi_0\, g(t,r). && (20)
\end{aligned}
$$

For $g_j(t,r) = 0$ along with g_4, and g_5 as given above, eqs. (15)–(18) reduce to:

$$
\begin{aligned}
\dot{x}h+\dot{x}g &= 2, && (21)\\
2\dot{h}+\dot{v}h+v'g &= 2, && (22)\\
e^{\nu}h'-e^{\lambda}\dot{g} &= 0, && (23)\\
2g'+\lambda'g+\dot{\lambda}h &= 2. && (24)
\end{aligned}
$$

Corresponding homothety vector eq. (6) in this case reduces to

$$H \;=\; g_4\frac{\partial}{\partial t}+g_5\frac{\partial}{\partial r}+\left(c_1\sin\varphi-c_2\cos\varphi\right)\frac{\partial}{\partial\vartheta}+\left(\cot\vartheta\left(c_1\cos\varphi+c_2\sin\varphi\right)+c_3\right)\frac{\partial}{\partial\varphi}. \tag{25}$$

As mentioned above, an attempt to solve eqs. (21) to eq. (24), with out imposing any restriction on stress-energy tensor T_{ab} or line element, produces a solution in the form of differential constraints. However, for $x(t,r)=2\ln r$ these differential constraints are reduced to meaningful expressions which then produce the metrics admitting the above mentioned homothety groups. For $x=2\ln r$, the line element eq. (5) comes out to be,

$$ds^2 \;=\; e^{\nu(t,r)}dt^2-e^{\lambda(t,r)}dr^2-r^2d\Omega^2, \tag{26}$$

and eqs. (21) to (24) yield,

$$\begin{aligned} g &= r, && (27)\\ 2\dot{h}+\dot{\nu}h+\nu' r &= 2, && (28)\\ h' &= 0,\quad e^{\nu}\neq 0, && (29)\\ \lambda' r+\dot{\lambda}h &= 0. && (30)\end{aligned}$$

From the eq. (30), we find

$$h\left(t,r\right)=-\frac{\lambda' r}{\dot{\lambda}}. \tag{31}$$

For $\dot{\lambda}\neq 0$, eq. (31) along with eq. (29) gives us

$$\left(\frac{\lambda' r}{\dot{\lambda}}\right)'=0,\quad \dot{\lambda}\neq 0,\quad \lambda=\lambda\left(t,r\right). \tag{32}$$

Here two cases arise, either $\dot{\lambda}=0$ or $\dot{\lambda}\neq 0$:

Case 1

For $\dot{\lambda}=0$, eq. (30) gives us $\lambda'=0$ as $r\neq 0$, which is possible only when $\lambda=0$. Thus $\nu=\nu(t,r)$ and $\lambda=0$ reduce eq. (26) to the metric

$$ds^2=e^{\nu(t,r)}dt^2-dr^2-r^2d\Omega^2, \tag{33}$$

that satisfies the eqs. (27), (29) and (30) with an additional constraint on $v\left(t,r\right)$ given by eq. (28), where $h=h\left(t\right)$. The corresponding homothety vector H, eq. (25) in this case reduces to

$$H=\phi_0\left(h\left(t\right)\frac{\partial}{\partial t}+r\frac{\partial}{\partial r}\right)+\left(c_1\sin\varphi-c_2\cos\varphi\right)\frac{\partial}{\partial\vartheta}+\left(\cot\vartheta\left(c_1\cos\varphi+c_2\sin\varphi\right)+c_3\right)\frac{\partial}{\partial\varphi}. \tag{34}$$

For example, the space-time eq. (33) for $\nu(t,r)=\frac{t}{r}$ and $h(t)=t$ reduces to

$$ds^2=e^{\frac{t}{r}}-dr^2-r^2d\Omega^2. \tag{35}$$

Above space-time eq. (35) satisfies the additional constraint eq. (28). In this case, the homothety vector H eq. (34) of the above space-time eq. (35) is given by

$$H=\phi_0\left(t\frac{\partial}{\partial t}+r\frac{\partial}{\partial r}\right)+\left(c_1\sin\varphi-c_2\cos\varphi\right)\frac{\partial}{\partial\vartheta}+\left(\cot\vartheta\left(c_1\cos\varphi+c_2\sin\varphi\right)+c_3\right)\frac{\partial}{\partial\varphi}, \tag{36}$$

clearly showing that the space-time eq. (35) admits four homotheties. For the space-time (35), Ricci scalar and the independent non-zero components of the Ricci tensor are:

$$\begin{aligned} &R_{00}=\frac{t^2}{4r^4}e^{t/r},\; R_{11}=-\frac{t}{r^4}(r+\frac{t}{4}),\\ &R_{22}=\frac{t}{2r},\; R_{33}=R_{22}\sin^2\theta,\; R=\frac{t^2}{2r^4}.\end{aligned} \tag{37}$$

Using EFEs eqs. (1) (without cosmological constant), the components of stress-energy tensor and the stress energy tensor T for the above space-time eq. (35) are given by:

$$\begin{aligned}
&\kappa T_{00} = 0,\, \kappa T_{11} = -\frac{t}{r^3},\, \kappa T_{22} = \frac{t(2r+t)}{4r^2},\\
&\kappa T_{33} = \kappa T_{22}\sin^2\theta, \kappa T = -\frac{t^2}{2r^4}.
\end{aligned} \tag{38}$$

Case 2

Now for $\dot{\lambda} \neq 0$, eq. (32) suggests that

$$\frac{\lambda' r}{\dot{\lambda}} = \alpha(t), \tag{39}$$

where $\alpha(t)$ depends on t only. Equation (31) along with eq. (39) yields

$$h(t,r) = -\alpha(t). \tag{40}$$

Substituting eq. (40) in eq. (28) yields

$$-2\dot{\alpha}(t) + \dot{v}h + v'r = 2. \tag{41}$$

Note that eq. (41) is separable for $\dot{\nu} = 0$. Thus, we may rewrite above equation as,

$$\dot{\alpha}(t) = \frac{v'r-2}{2},\ \ \dot{\nu} = 0. \tag{42}$$

In the above eq. (42), the $L.H.S.$ is function of time t only whereas $R.H.S.$ is function of r only, which is possible only when,

$$\dot{\alpha}(t) = \frac{\nu' r - 2}{2} = \alpha_1, \tag{43}$$

where α_1 is separation constant and the above eq. (43) can be solved by separating it into two parts. We get

$$\alpha(t) = \alpha_1 t + \alpha_2, \tag{44}$$

where α_2 is a constant of integration and,

$$\nu(r) = 2(\alpha_1 + 1)\ln r. \tag{45}$$

The eqs. (19) and (20) along with eqs. (27), (40) and (44) reduce to

$$g_4 = -\phi_0\left(\alpha_1 t + \alpha_2\right), g_5 = \phi_0 r. \tag{46}$$

We find now $\lambda(t,r)$. The eqs. (39) and (44) together may be written as,

$$\lambda' r = \dot{\lambda}(\alpha_1 t + \alpha_2). \tag{47}$$

By the same argument used for eq. (43), eq. (47) is possible only when

$$\lambda' r = \dot{\lambda}(\alpha_1 t + \alpha_2) = m, \tag{48}$$

where m is a constant. Thus, from eq. (48) we get

$$\lambda(r) = m\ln r \ \ \text{and} \ \ \lambda(t) = \frac{m}{\alpha_1}\ln\left(\alpha_1 t + \alpha_2\right). \tag{49}$$

The eq. (49) implies that,

$$\lambda(t,r) \ = \ m\ln\ r + \frac{m}{\alpha_1}\ln(\alpha_1 t + \alpha_2),\ \alpha_1 \ \neq\ 0. \tag{50}$$

The eq. (26) along with (45) and (50) is given by the following metric,

$$ds^2 = r^{2(\alpha_1+1)}dt^2 - r^m(\alpha_1 t+\alpha_2)^{m/\alpha_1}\,dr^2 - r^2 d\Omega^2. \tag{51}$$

Thus the corresponding homothety vector H, eq. (25) is given by,

$$H = -\phi_0\left(\alpha_1 t+\alpha_2\right)\frac{\partial}{\partial t} + (\phi_0 r)\frac{\partial}{\partial r} + (c_1\sin\varphi - c_2\cos\varphi)\frac{\partial}{\partial\vartheta} + (\cot\vartheta\,(c_1\cos\varphi + c_2\sin\varphi)+c_3)\frac{\partial}{\partial\varphi}. \tag{52}$$

Here the arbitrary constants are ϕ_0, c_1, c_2, c_3 whereas α_1 and α_2 are constants on metric. So that the generators of homothety group H_4 are,

$$X^0 = -\left(\alpha_1 t+\alpha_2\right)\frac{\partial}{\partial t} + r\frac{\partial}{\partial r}, \tag{53}$$

$$X^1 = \sin\varphi\frac{\partial}{\partial\vartheta} + \cot\vartheta\cos\varphi\frac{\partial}{\partial\varphi}, \tag{54}$$

$$X^2 = -\cos\varphi\frac{\partial}{\partial\vartheta} + \cot\vartheta\sin\varphi\frac{\partial}{\partial\varphi}, \tag{55}$$

$$X^3 = \frac{\partial}{\partial\varphi}, \tag{56}$$

where $[X^0, X^i] = 0$ and $[X^i, X^j] = X^k$ for $i.j,k = 0,1,2,3, i \neq j \neq k$. Showing that there are four homothety vectors given by eqs. (53)-(56). For the space-time eq. (51), the independent non-zero components of the Ricci tensor and the Ricci scalar are:

$$\begin{aligned}
R_{00} &= \alpha^{-2}[-1+\frac{m\alpha_1}{2} + (1+\alpha_1)(3r^{-2(\alpha_1-1)}\alpha^{\frac{2(\alpha_1-1)}{\alpha_1}} - r^{2\alpha_1-1}(1+\alpha_1)\alpha^{\frac{2\alpha_1-1}{\alpha_1}} + \\
&\quad r^{2\alpha_1-m}(1-m+2\alpha_1)\alpha^{\frac{2\alpha_1-m}{\alpha_1}})],\\
R_{01} &= \frac{2}{r\alpha},\\
R_{11} &= r^{-2}[2+2(1+\alpha_1)-(1+\alpha_1)^2 - r^{-2(\alpha_1-1)}\alpha^{\frac{-2(\alpha_1-1)}{\alpha_1}} + \frac{1}{2}mr^{m-2\alpha_1}(m-\alpha_1)\alpha^{\frac{m-2\alpha_1}{\alpha_1}}],\\
R_{22} &= r^{-3}[r^3 - r\alpha_1\alpha^{\frac{-2}{\alpha_1}} + (m-1)r^{3-m}\alpha^{\frac{-m}{\alpha_1}} + \alpha^{\frac{-3}{\alpha_1}}(1-3r\alpha^{\frac{1}{\alpha_1}})],\\
R_{33} &= \left(R_{22} + \frac{1-r\alpha^{\alpha_1^{-1}}}{r^4\alpha^{4\alpha_1^{-1}}}\right)\sin^2\theta,\\
R &= r^{-6}[-\frac{1}{2}m^2r^{4-2\alpha_1}\alpha^{-2} + \alpha^{-\frac{m}{\alpha_1}}r^{4-m}(3\alpha_1^2-3m-\alpha_1 m+3\alpha_1) + \alpha_1 m r^{4-2\alpha_1}\alpha^{-2} + r^{-2\alpha_1-m+6}\\
&\quad \alpha^{\frac{-2\alpha_1-m+2}{\alpha_1}} - r^{4-2\alpha_1}\alpha^{-2} - 2r^4 - r^3\alpha^{-\frac{1}{\alpha_1}}(\alpha_1^2+2\alpha_1+1) + r^2\alpha^{-\frac{2}{\alpha_1}}(5\alpha_1+9) - r\alpha^{-\frac{3}{\alpha_1}} - \alpha^{-\frac{4}{\alpha_1}}].
\end{aligned} \tag{57}$$

The stress-energy tensor for the above space-time eq. (35) from EFEs eqs. (1) (without cosmological constant), comes out to be:

$$\begin{aligned}
\kappa T_{00} = \ &\frac{1}{4}[(m^2-2)\alpha^2 - 2\alpha^{\frac{2-m}{\alpha_1}-2}r^{2-m} + \alpha^{-\frac{m+2}{\alpha_1}}(2\alpha^{\frac{2}{\alpha_1}}r^2({\alpha_1}^2 - m\alpha_1 + 3\alpha_1 + m+2) - 2\alpha^{\frac{m}{\alpha_1}}\\
&r^m(\alpha^{\frac{1}{\alpha_1}}(\alpha_1+1)^2 r - \alpha_1))r^{2\alpha_1-m-2} + 2\alpha^{-\frac{4}{\alpha_1}}(2\alpha^{\frac{4}{\alpha_1}}r^4 - 3\alpha^{\frac{2}{\alpha_1}}r^2 + \alpha^{\frac{1}{\alpha_1}}r+1)r^{2\alpha_1-4}],
\end{aligned} \tag{58}$$

$$\kappa T_{01} = \frac{2}{\alpha r}, \tag{59}$$

$$\begin{aligned}
\kappa T_{11} = \ &\frac{1}{4}[(m^2-2)\alpha^{\frac{m-2\alpha_1}{\alpha_1}}r^{-2\alpha_1+m-2} - 2\alpha^{\frac{m-4}{\alpha_1}}r^{m-6} - 2\alpha^{\frac{m-3}{\alpha_1}}r^{m-5} + 18r^{m-4}\alpha^{\frac{m-2}{\alpha_1}} -\\
&2\alpha^{\frac{m-1}{\alpha_1}}r^{m-3} - 4\alpha^{\frac{m}{\alpha_1}}r^{m-2} + 2\alpha_1^2 r^{-3}(r - \alpha^{\frac{m-1}{\alpha_1}}r^m) - 6r^{-2}(m-2) - 2\alpha_1\\
&(2\alpha^{\frac{m-1}{\alpha_1}}r^{m+1} - 5\alpha^{\frac{m-2}{\alpha_1}}r^m + mr^2 - 3r^2)r^{-4} - 2\alpha^{\frac{2}{\alpha_1}-2}r^{-2\alpha_1}],
\end{aligned} \tag{60}$$

$$\begin{aligned}
\kappa T_{22} = \ &\frac{1}{2}[3r^{-m}\alpha^{-\frac{m}{\alpha_1}} + \alpha^{-\frac{2}{\alpha_1}}r^{-2}(3-2r\alpha^{\frac{1}{\alpha_1}}) + m(r^{-2\alpha_1}\alpha^{-2} - r^{-m}\alpha^{-\frac{m}{\alpha_1}}) - 2^{-1}(2+m^2)\\
&r^{-2\alpha_1}\alpha^{-2} + r^{2-m-2\alpha_1}\alpha^{-2+\frac{2-m}{\alpha_1}} - \alpha^{-\frac{4}{\alpha_1}}(-1+r\alpha^{\frac{1}{\alpha_1}} + 3r^2\alpha^{\frac{2}{\alpha_1}} - r^3\alpha^{\frac{3}{\alpha_1}})r^{-4} -\\
&(2+m)r^{-m}\alpha^{-\frac{m}{\alpha_1}} + \alpha_1 - (\alpha^{-\frac{1}{\alpha_1}}r^{-1} - 3r^{-m}\alpha^{-\frac{m}{\alpha_1}})\alpha_1^2],
\end{aligned} \tag{61}$$

$$\kappa T_{33} = \left[\kappa T_{22} + \alpha^{-\frac{4}{\alpha_1}} r^{-4}\left(1 - r\alpha^{\frac{1}{\alpha_1}}\right)\right]\sin^2\theta, \tag{62}$$

$$\kappa T = r^{-6}[\frac{1}{2}m^2 r^{4-2\alpha_1}\alpha^{-2} - \alpha^{-\frac{m}{\alpha_1}} r^{4-m}(3\alpha_1^2 - 3m - \alpha_1 m + 3\alpha_1) - \alpha_1 m r^{4-2\alpha_1}\alpha^{-2} - r^{-2\alpha_1-m+6}\alpha^{\frac{-2\alpha_1-m+2}{\alpha_1}} + r^{4-2\alpha_1}\alpha^{-2} + 2r^4 + r^3\alpha^{-\frac{1}{\alpha_1}}(\alpha_1^2 + 2\alpha_1 + 1) - r^2\alpha^{-\frac{2}{\alpha_1}} (5\alpha_1 + 9) + r\alpha^{-\frac{3}{\alpha_1}} + \alpha^{-\frac{4}{\alpha_1}}]. \tag{63}$$

III. CONCLUSIONS

We have found the homotheties and corresponding metrics for a class of spherically symmetric space-times (5) to admit G_3 as the maximal group of motions for $x(t,r) = 2\ln r$. The motivation behind is the classification of spherically symmetric space-times according to their homotheties without imposing any restriction on the stress-energy tensor. The homotheties and the corresponding metrics are already known for the space-times admitting G_4, G_6 and G_{10} as maximal isometry groups whereas for the space-times admitting G_3 as the isometry group the solution is known in the form of differential constraints which needs further consideration. We found the homotheties and the corresponding metrics admitting G_3 as the maximal group of isometries for a class of spherically symmetric space-times (5), as mentioned above. For a subclass of spherically symmetric space-times, for which $\dot{\lambda} = 0$, the metric is given by eq. (33) and the corresponding homothety vector is given by eq. (34) subject to the additional constraint eq. (28) in terms of derivatives of the metric coefficients. In particular for $\nu(t,r) = \frac{t}{r}$, $h(t) = t$, the metric satisfying the additional constraint eq. (28) and the corresponding homothety vector are given by the eqs. (35) and (36) respectively. Whereas, for $\dot{\lambda} \neq 0, \dot{\nu} = 0$ the metric eq. (5) reduces to the metric eq. (51) and the corresponding homothety vector is eq. (52). Stress-energy tensor for the above space-times are given by the eqs. (38) and (58)- (63). It might be interesting to see these results according to different types of stress-energy tensor.

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